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ABSTRACT

This paper examines students' mathematics learning in a social context as two versions of an instructional sequence designed to facilitate students' development of a conceptual foundation for place-value numeration are enacted in two third grade classrooms. The same instructional sequence is used in each classroom with the regular classroom teacher conducting the instruction in one class and an expert graduate student conducting instruction in the other class. Other differences between the two teaching experiments are that in the second situation, there is a deliberate focus on notation and symbolizing as well as a series of computer microworlds which are used to facilitate imagery of the instructional scenario of a candy factory. Analysis of the data indicates that the candy factory imagery both supports and constrains the mathematical conceptions the children develop. A critical difference between the two teaching experiments is the manner in which the use of imagery is constituted. The use of the computer microworlds enables more children to participate in a meaningful way in the activities and their understanding of place-value is enhanced as a result. Contains 19 references. (Author/DDR)

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# COMPARING INDIVIDUAL AND COLLECTIVE MATHEMATICAL ACTIVITY ACROSS TWO TEACHING EXPERIMENTS

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The purpose of this paper is to examine students' mathematics learning in social context as two versions of an instructional sequence designed to facilitate students' development of a conceptual foundation for place-value numeration were enacted in two third-grade classrooms. In each classroom, an instructional sequence, called the Candy Factory, was the subject of a six to eight week long classroom teaching experiment. In the first of the two teaching experiments, the regular classroom teacher conducted the instruction. In the second, an expert graduate student was the teacher. Further differences between the teaching experiments are that the second deliberately set out to focus on the role of notation and symbolizing while the first did not. In addition, a series of McCandy Factory computer microworlds was used in the second teaching experiment as a means of facilitating the imagery of the candy factory scenario when large quantities were involved while in the first classroom, overhead projector materials and drawings were the primary means used to develop imagery.

Analyses of the students' learning in both classrooms show that the candy factory imagery supported and constrained the mathematical conceptions the children developed. However, the way in which use of the imagery was constituted differed in the two classrooms. It is here that the means of symbolizing and notating played a critical role. In both classrooms pictures and drawings were used prior to abstract numeric symbols. However, the dynamic nature of the McCandy Factory microworlds provided a richer imagistic basis that the use of drawings and diagrams alone. Consequently, it was possible for more children to participate in a meaningful way in discussions of complex problems involving three-digit addition and subtraction. As a result, their understanding of place-value numeration was enhanced. Examples are used to document the nature of the children's learning in each case and to clarify the differences in the types of arguments and explanations that developed as normative in the two classrooms.

## Theoretical Framework

The theoretical framework used to guide the development and research of the Candy Factory instructional sequence is compatible with the Freudenthal Institute's process of developmental research (Gravemeijer, 1994). The underlying assumption suggests that instructional design and classroom-based research are reflexively related in that each phase informs the other. The instructional design phase involves developing materials that are compatible with students' current ways of knowing but that are also able to support their development of more sophisticated ways of notating and organizing their activities. These ways of notating can later become the basis for students' more formal mathematical activity. In this way, the materials are described as bottom-up in nature because the students' initial activity can serve as an imagistic basis to support their development of more formal activity (Cobb, Gravemeijer, Yackel, McClain, & Whitenack, in press).

The analytic framework guiding the research phase of the developmental research cycle follows the emergent perspective described by Cobb and Yackel (1996) in which psychological and sociological perspectives on learning are coordinated to describe learning as an active process in which students reorganize their own beliefs as they interact with others in the social setting of the classroom. Examples from the data are used to illustrate the processes by which students' own developing conceptions are supported and implicitly encouraged by their participation in whole-class discussions. Further, the evolution of ways of symbolizing in these whole-class and individual reflection processes is documented for each of the classrooms studied. This is accomplished by examining the taken-as-shared classroom mathematical practices surrounding the negotiation of symbolic meaning and the way in which these supported and constrained the individual and collective ways of knowing. This approach to analysis differs from traditional cognitive studies that attempt to isolate the influence of particular instructional variables such as the use of symbols or computer software whose meaning is assumed to be self-evident. Instead, ways of

symbolizing as aspects of classroom mathematical practices are considered to support the emergence of mathematical meaning in the classroom (Cobb, Gravemeijer, et al., in press).

## Method and Data

### Data

The data for this paper are taken from two third-grade classroom teaching experiments in mathematics. Each teaching experiment was from six to nine weeks in duration. All mathematics lessons were videotaped for later analysis. Video recordings include whole-class activities and the activity of selected students during individual and small-group classroom work. In addition, individual interviews were conducted both before and after the teaching experiments with all of the students in one class and with a representative sample of students in the other class. Interviews of pairs of students were also conducted for the class that made use of the computer. In the latter interviews, students were asked to solve problems both with and without the computer. The data corpus consists of sets of videotaped recordings, field notes, and copies of student work from all lessons and interviews at each of the two sites.

### The Candy Factory Instructional Sequence

The Candy Factory sequence engages students in activities in which they imagine working in a candy factory that packages candies in rolls containing ten pieces, in boxes containing ten rolls, and in cases containing ten boxes (Cobb, Yackel, & Wood, 1992). Instructional activities include asking students to estimate and count large quantities of candies, figure out different ways a given quantity of candies might be packaged, and carry out transactions that involve adding candies to the factory storeroom or sending out candies to fill an order. The intent of these activities is to facilitate students' development of an imagistic basis for packing or unpacking as conserving quantity. For example, if an order comes in requiring a large number of rolls of candy, then students may need to "unpack" a box to fill the order. In this way the conventional classroom practice of "trading" in multi-digit subtraction becomes grounded in the students' real-world imagery of unpacking a box

or a roll in the candy factory and hence has meaning. In later stages of the instructional sequence, an inventory form is introduced as a way to record, first, quantity of candies in the factory storeroom, and later, transactions that take place in the storeroom when more candies are brought in from the factory or candies are sent out to fill an order. This form is useful to develop a conceptual basis for conventional notation for adding and subtracting multi-digit numbers. Initially, the form is a model of the students packing and packing activity and later becomes a model for their more formal arithmetical reasoning with multi-digit numerals.

Research Approach

The type of action research that forms the basis for these teaching experiments involves active collaboration between the teacher and the researchers based on daily observations and reflections. In this way, the researchers are actively involved in the teaching, and the teacher is actively involved in the research. On a daily basis the project staff analyzed the nature of the students' mathematical activity and noted which aspects of the instructional sequence were helpful to students, and which were problematic. In each teaching experiment, these observations were used to inform the revisions of the activity sequences throughout. For example, using static notation to notate the dynamic activity of transactions in the candy factory as well as the transformations of packing and unpacking was challenging for the teachers as well as for the students. Further, using physical materials to actually carry out transformations and transactions was problematic because of the large quantities involved. For these reasons, and others, researchers at the second site developed a set of computer microworlds to support students' efforts with the candy factory scenario and to address some of the notational issues. These microworlds were used by students in collaborative small-group settings to carry out transformations and transactions in the candy factory and by the teacher in whole-class discussions as a means of linking students' descriptions of their transactions and transformations to activity in the candy factory. Again, the research teams (which included the teachers) noted the nature and quality of the students' discussions in order to get a sense of how their thinking about place value

and three-digit addition and subtraction developed over the course of the teaching experiment.

The MacCandy Factory Microworlds

The microworlds developed to accompany the Candy Factory instructional sequence contain linked representation systems that indicate multiple ways of representing activity in the candy factory. This design is based on the work of Thompson (1992) who created the *Blocks* microworld. This latter program, which contains icons that depict Diene's blocks linked to place-value notation, was designed to enable students to manipulate numeric symbols in order to effect desired changes on the blocks. For example, if a student wanted to "explode" a "flat", he would click on the numeral indicating the number of "flats." In this way, the linked representation system supported students' sense-making activities because they were able to utilize their imagery of how to act on the blocks to guide their activity with mathematical symbols. Analogously, when students were acting with the MacCandy Factory microworlds, they could act with the numeric symbols to effect changes on the icons representing boxes, rolls, and pieces, or they could act on the icons to effect changes on the numerals.

## Overview of Results

Analysis of the data from both sites focuses on the ways the individual students developed personally meaningful ways of acting and communicating their actions as they participated in the social and mathematical practices established in their classroom microcultures. In particular, the analysis focus on two issues: 1) the ways in which the students' (and the teacher's) ways of symbolizing supported and constrained their thinking and communication; and 2) the nature of the arguments the students gave to explain and justify their solutions and thinking. The overall goal is to address these two issues by exploring the ways of explaining and the evolution of taken-as-shared mathematical practices that evolved in the two different settings.

Symbols

At both sites, the means of symbolizing became an explicit topic of conversation. The students' views of the purpose of symbolizing evolved over time from communicating activity in the candy factory to describing how actions on the graphics had quantitative significance that had to be indicated with the numeric symbols. The students' symbolizing activity gave explicit indications of the ways in which they were able to reify and consequently build on their initially informal actions. For example, when the candy factory notation was used to support whole-class discussions, the type of language shifted from abstract notions of "trading" to more concrete descriptions of "packing." Over time, this shift in language reflected the students' shift in viewing the icons on the screen as boxes and rolls to more sophisticated conceptual units. This indicates that students may have been able to reify their activity of packing and unpacking to develop increasingly sophisticated conceptions of place-value numeration based on their own interpretations of the symbols they used to represent their goal-based activities.

Explanations and Justifications

Analyses of the nature of the discussions at both sites reflect the view that the imagery children developed for the candy factory scenario both supported and constrained their mathematical explanations and justifications. For example, one issue that received a great deal of discussion at the second site was how to indicate that a roll had been packed. The computer version of the inventory form increments the active number of rolls by one, but does not indicate a "1" as being "carried" as would conventional school notation. Despite this, the children knew that the "carry" notation was deemed important from their knowledge of the standard algorithm, and developed arguments based in the imagery of the candy factory to rationalize their adding the one in the rolls' column. For example, one student who was justifying the use of the cross-out notation rather than simply putting a "1" in the rolls' column argued that the storeroom manager needed to be able to look at the inventory form and know that there were initially 11 candies and that a roll was subsequently packed. These conversations illustrate the students' growing understandings of the inherent

conventions of the standard notation and their increasingly sophisticated conceptions of place value. This type of imagery-based argument exemplifies the degree to which the students' discourse was conceptual rather than computational in nature (Thompson, Philipp, Thompson, & Boyd, 1994).

## Analysis

In order to analyze the reflexivity between students conceptions and their participation in their classroom mathematical cultures, we first discuss the interactive constitution of the pedagogical symbols. Next we discuss the role of different instructional materials in supporting whole-class discourse. In this section we describe various instructional materials that were used in the two teaching experiments and explain how they constrained and supported both the justifications that students gave and their understanding of what constituted an acceptable justification. In the remainder of the paper we discuss the evolution of taken-as-shared classroom mathematical practices as the instructional sequence progressed. We conclude this last section with a detailed analysis of two children in an interview setting solving a problem with the support of the MacCandy Factory microworld. This analysis explicates the relationship between the children's interpretations of their activity with the dynamic symbols and their numerical reasoning.

### The Interactive Constitution Of Pedagogical Symbols

In this section of the paper we focus on the taken-as-shared meanings that emerged for the candy factory scenario and accompanying pedagogical symbols and on the process by which those meanings emerged. These were strongly influenced by the individual children's understandings of quantity and of ten, which varied considerably. For example, in interviews conducted just prior to the instructional sequence only some of the children in each of the two classes demonstrated the ability to coordinate units of ten and one (Steffe, Cobb, & von Glasersfeld, 1988). Others solved problems using approaches that suggested their concept was ten as an abstract singleton (Cobb & Wheatley, 1988), that is, they treated

ten as a large abstract unit but, for them, it was not necessarily composed of ten individual units.

In each class, on the first day of the instructional sequence the teacher attempted to establish the notion of packing candies into rolls, boxes, and cases. After an initial discussion in which the students concluded that packing made it easier to count and keep track of large numbers of candies, the children engaged in actually packing up quantities of candies. Unifix cubes were used as "pretend" candies and were packed into bars of ten (rolls) and bags of ten rolls each (boxes). In subsequent lessons each class used drawings to show various quantities of candies. The intention was that drawings would serve as pedagogical symbols which would support the children's developing understanding of place-value numeration. The drawings might be used both to pose tasks and to record results. On one hand, the drawings might be seen as substitutes for manipulative materials, such as Unifix cubes. On the other hand, they might be seen as an alternative to specifying the quantity numerically. However, it became apparent that the drawings were not unambiguous representations. The drawings did not *carry* meaning. Instead, each student necessarily had to interpret the drawings and did so in terms of his or her own understandings while participating in the instructional tasks. For this reason it was critical that the classes discuss the children's interpretations so that taken-as-shared meanings could develop. Only then did the drawings become a useful means of communication among the members of the classes. However, it was as taken-as-shared meanings emerged that the children's own understandings advanced. This illustrates the reflexive relationship between taken-as-shared mathematical practices and individual children's mathematical understandings (Cobb & Yackel, 1996).

It might be argued that the meanings of the drawings were fixed by the conventions of the candy factory. A box necessarily indicates one hundred candies (or ten rolls) and a roll necessarily indicates 10 candies. Nevertheless, children's conceptions of number and of ten played heavily into their interpretations of the drawings. As a result, children in the same

classroom did not all agree on the quantity indicated by drawings of boxes, rolls, and individual candies. For example, on the second day of the instructional sequence at the first site the teacher showed a drawing of three boxes, four rolls, and five pieces on the overhead screen. One child, Teri, reported that she saw 345 candies, but another student, Jon, reported that he saw 75. Teri's description indicates that she was able to conceptualize a box as either one box, ten rolls, or 100 pieces. Further, the way she incremented as she continued her explanation indicates that she was able to think about the number of items (boxes, rolls, or pieces) and the quantity they represent interchangeably. We infer from this that her understanding of number was at least at the level of composite unit (Steffe, 1992).

Teri: 345, because I saw three boxes and there's 10 rolls in each box and that's 100 and that was 300. And four rolls with 10 and that was 340 and I saw five loose pieces and that was 345.

By contrast, Jon did not distinguish the ten units that comprise a box (rolls) from the ten units that comprise a roll (pieces).

Jon: I saw three boxes, three boxes of ten. Ten, 20, 30 and I saw four tens out of the boxes. And I said 40, 50, 60, 70 and the five. Seventy, 71, 72, 73, 74, 75.

Our inference is that his was not a simple misunderstanding of the task but indicated his inability to coordinate units of different rank. In the ensuing discussion five different students participated with the teacher in attempting to clarify why Jon's interpretation was inconsistent with the conventions of the symbols. These students' comments reflected their own interpretations and understandings. As the discussion progressed the teacher wrote the numeral 100 over each box, the numeral 40 under the four rolls and the numeral 5 under the five individual pieces to help clarify the children's explanations. Then Jon was able to state that the total quantity in the drawing was 345. However, we would not claim that he was able to interpret the drawing in the same way as Teri.

It was through repeated discussions such as these that taken-as-shared meanings gradually emerged for the drawings of boxes, rolls, and candies and thus for the candy factory scenario itself. From the interactionist perspective, it is not possible to superimpose meanings by simply "telling" the students. According to this perspective the source of meaning is not intrinsic, merely to be apprehended. Instead, meaning arises "in the process of interaction between people. The meaning of a thing for a person grows out of the ways in which other persons act toward the person with regard to the thing. ... [T]he use of meanings by the actor occurs through a process of interpretation" (Blumer, 1969, pp. 4-5). In our case all of the students had participated in the development of the packing scenario when the conventions of packing were specified. Nevertheless, a number of them were not able to "read" a quantity directly from a picture display. Their interpretations reflected their current conceptions, including their capabilities to coordinate units and their understandings of the packing scenario. It is apparent that interpreting a drawing showing various quantities of boxes, rolls, and pieces was a conceptual task.

While the drawings might be thought of as iconic representations that could engender the imagery of packing and unpacking activity, the research team discussed two inherent limitations. One is that there is nothing in the drawings themselves to confront an interpretation such as the one Jon gave. This was particularly important when children worked individually or in pairs. For example, when working with their drawings children might "unpack" a box to get (only) one roll without experiencing a conflict of the type that would develop if they were actually unpacking using physical materials, such as Unifix cubes. In this case, class discussions had to bear the entire weight of countering such interpretations. A second limitation of the drawings is that, as introduced, they were static in nature. Thus, a drawing indicated the state of the candy factory at any given time. To indicate the activity of packing or unpacking (or of transactions, which were introduced later in the instructional sequence) some means needed to be developed to indicate action. For these reasons, in the first classroom some tasks were posed on the overhead projector using

transparent flats, longs, and cubes especially designed for overhead use. The flats and longs are scored to show the individual units and are designed so that they can be decomposed or combined. Thus, the activity of packing or unpacking can actually be carried out and viewed on the overhead screen. (Nevertheless, a record that traces the action does not result.) At the same time students continued to use drawings for paper and pencil tasks. In the second classroom, the accommodation that was made was to use a series of MacCandy Factory computer microworlds (Bowers, 1995, 1996) in which children could "pack" or "unpack" candies, rolls and boxes by clicking on icons. The MacCandy Factory microworlds have the added feature of recording and displaying packing and unpacking activity. Thus, the computer screen depicts an initial configuration of candies, the final configuration after transformations (packing and unpacking) and transactions (adding and subtracting), and a trace of the activity.

In the next section of the paper we discuss the role of these different instructional materials in supporting children's learning in the whole-class setting.

The Role of Different Instructional Materials in Supporting Whole-Class Discourse

One of the instructional goals of the candy factory sequence was to support students' development of increasingly sophisticated notions of place value. To this end, many of the activities centered around the imagery of packing and unpacking. During class discussions of these activities, the issue of how these transformations preserve quantity was often an explicit topic of conversation. This section examines three classroom episodes. The first episode, which is taken from the first teaching experiment, is used to clarify how the use of overhead materials further contributed to the interactive constitution of a taken-as-shared meaning for the drawings and for the candy factory scenario itself, especially for packing and unpacking activity. The second and third episodes are taken from the second classroom teaching experiment. They examine the ways in which the concept of preserving quantity emerged as the students discussed and justified different ways that a collection of candy might be found in the storeroom. In the second episode, the teacher notated the students'

suggestions using a chart of drawings written on a white board. In the third episode, the teacher notated the students' suggestion with the dynamic symbols in the MacCandy factory.

Episode 1: Quantification Using Overhead Materials. The teacher introduced the overhead materials by explicitly telling the students that she would use the flats, longs, and cubes to represent boxes, rolls, and pieces of candy, respectively, and discussed how these materials differed in appearance from the drawings she had made previously. She then initiated a discussion of the relationship between the rolls, boxes, and pieces and led the class in counting the number of individual pieces in a box. Subsequently she displayed a configuration of flats, longs, and cubes on the overhead screen and asked, "Okay, how many think they know how many pieces of candy were up there? ... How many pieces of candy do you see?" In posing the question this way the teacher encouraged the children to think in terms of the imagery of the candy factory. This subtle teacher move was critical in generating responses that might be productive for helping students relate the individual units, units of ten and units of hundreds through the imagery of packing and unpacking.

For example, the teacher posed one flat, three groups of four longs each, and nine cubes as shown in Figure 1. The first child to respond gave 229 as her answer and explained,

Student: First I saw the box of candy and it had 100 in it and I knew it was 100 and then I saw four, 40 beside of it and then I saw 40 more and I added it up and I got 80 and then I saw 40 more on the side of it so I added. I added one, two tens to that one and I got another hundred so I added the two hundred up and I got 200 and I knew 40 take away two tens, 40 take away two tens is 20 and I saw (inaudible) and so I saw 3, 6, 9, and I added it to the 20 and I added up 229.

Another child gave a similar explanation but referred to the hundred that was created by combining 40, 40 and 20 as "another box." The teacher capitalized on this contribution.

Teacher: Okay, let's look on our overhead and see how—A lot of you mentioned you made another box. Let's see if we can make another box. How could we make another box here? Which would you start with, Teri?

She then created a flat by moving ten of the longs together following Teri's direction.

Teri: I would start with that 40 (on the left) and then put that other 40 (in the center) with it. And that will be 80. And then take two from that 40 right here (on the right) and that will be 100, and that'll be a box. And you will have 20 left and you will be [have] 9 left and it's [it'd] be 229.

After the class counted the ten rolls that formed the box, by tens, to get 100, the teacher summarized the discussion.

Teacher: Okay, so you were able then to see, a lot of you, that those loose rolls, when you package [them] into another box you can count them and see right away that our number here is 229 that you had.

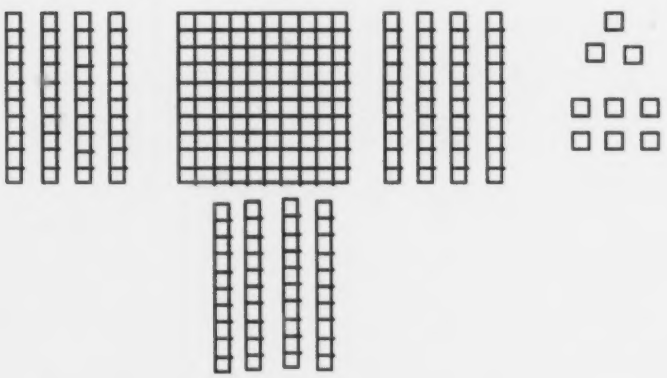

Figure 1. Overhead display of 229 using flats, longs, and cubes.

The teacher's last comment indicates her own expectation that the students would be able to easily specify the quantity of a configuration that is in canonical form, that is, a configuration that is maximally packed. As the earlier example showed, not all of the

children could do so initially. The data show that as the instructional sequence progressed children were increasingly able to do so. For example, by the sixth lesson a majority of the children routinely drew (correct) canonical representations of quantities that were described numerically in problem situations. We would argue that it is through participating in class activities and discussions such as these that children develop both the relevant conceptual understanding of number and of the conventions of the pedagogical symbols. As a consequence, children no longer had to justify their interpretations of the drawings. They could assume that others "saw" the same quantities in drawings of boxes, rolls and candies. These became taken-as-shared classroom mathematical practices (Cobb & Yackel, 1996).

In the next two episodes, we investigate the different mathematical practices that arose during whole-class discussions that were supported by a) a chart containing iconic drawings and b) the MacCandy Factory microworld. This analysis focuses on the different ways for justifying answers that emerged during the conversations, and how these norms affected consequent student interpretations. The focus here is on normative understandings of what constitutes mathematical justification. In the two episodes that follow, the taken-as-shared methods of justification are qualitatively different. The analysis explicates how the students' mathematical conceptions and the MacCandy Factory microworlds supported justifications that evolved from imagining packing up to discussing how transformations preserve quantity.

Episode 2: Different Ways with Chart of Iconic Drawings. The objective of this activity was to discuss the different ways that 532 candies might be found in the storeroom. The conversation began when one student, Lenore, stated that she drew 5 boxes, 3 rolls, and 2 pieces. The teacher drew this suggestion on the whiteboard chart with small boxes, lines, and circles, and then asked for someone to check the answer. A second student, Andrea, stated, "We know that a box is 100 so you have 5 in the hundreds so that is 500, the rolls each have ten so that is 30, plus 500 is 530, then plus 2 is 532." Given that neither the teacher nor any of the students questioned this justification, it seems reasonable to suggest

that the practice of justifying answers by enumerating collections appeared to be taken-as-shared.

Next, another student, Martine, suggested 500 rolls and 32 pieces. When asked to check this one, several students justified their solutions by imagining unpacking or packing. For example, one student said, "If it was 500 rolls, that would be 5000 pieces." Another said she thought that Martine meant 5 rolls. A third student, Linda, described her solution in terms of imagining packing the 500 pieces to make 5 boxes.

Linda: Well, if you subtract 100, if you subtract 100 pieces from that and make 400 and then you could make that a box. And then if you subtract 100 more pieces and that would make another box, and then that would be 300. And then you subtract another hundred, and that would be another box and that would be 200, and then another hundred would be another box.

After several others spoke, Crystal, initiated the following exchange:

Crystal: It couldn't be five rolls because there's ten in a roll and that would be 50.

Teacher: OK, Crystal says that if we have 5 rolls, that would be 50 pieces. (writes on board "5 rolls = 50 pieces" ) so that would not be enough. Martine?

Martine: I'll change it to 50.

Teacher: All right, you are changing it to 50, why are you changing it to 50?

Martine: Well, if I have 5 rolls, that would be 50 and I want to send out rolls to be 50.

The teacher redescribed Martine's explanation in terms of the imagery of acting in the candy factory by drawing 50 rolls and packing them as shown in Figure 2.

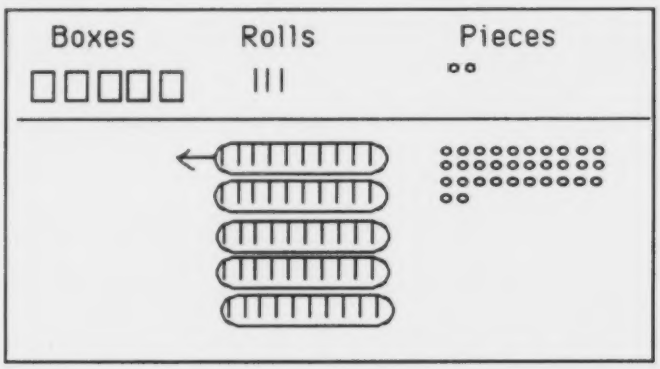


Figure 2. Teacher's tally marks showing the packing of 50 rolls.

The question of why Martine was convinced to change his interpretation of his answer is unclear. However, as Figure 3 shows, he actually had drawn 50 rolls on his paper. For him, these rolls appeared to be numerical composites rather than composite units (Steffe, 1992). That is, they were each ten candies, not one ten–so therefore he interpreted them as 500 candies in rolls.

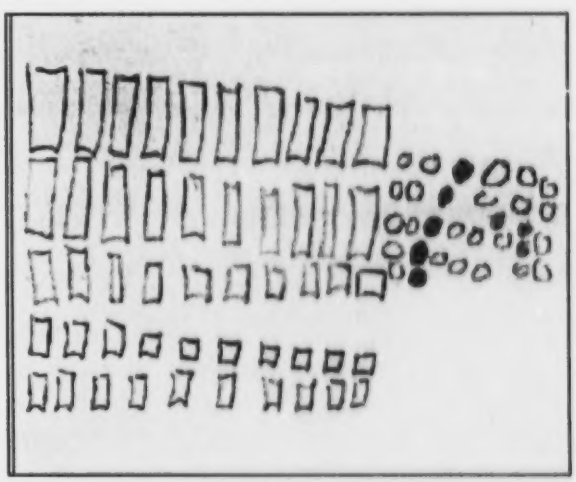

Figure 3. Scanned image of Martine's work showing that he actually drew 50 rolls, but interpreted his work as 500 rolls.

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From a psychological perspective, this episode suggests that students such as Martine were thinking about their activity differently than students who made interpretations such as Linda's. Linda's action imagery indicates that, for her, the activity of packing had become reified. Additionally, the way she viewed the pieces contained in the boxes can be described in terms of part-whole relations (Steffe, 1992). Her description of packing 5 boxes in succession may have also supported other students, such as Martine, for whom the quantities were numerical composites.

From an interactionist perspective, the discourse pattern that occurred during this episode involved having one student describe a different *product of acting* with the candies, and then having another student justify this product using action imagery to imagine packing up to arrive at the mathematical quantity of 532. In contrast, in the next episode, which occurred several weeks later, the students began each "different way" by describing the *process of acting* that was used to create "the way." This shift eliminated the need to carry out the justification because each action was implicitly justified as a transformation of the last.

Episode 3: Different Ways with MacCandy Factory. The instructional activity during this segment was to discuss different ways to have 316 candies in the storeroom. This whole-class discussion was supported by the MacCandy factory microworld, which was projected onto a screen using an LCD panel. As in the episode described above, the first student to answer the question suggested the canonical form, 3 boxes, 1 roll, and 6 pieces. Because this was a taken-as-shared interpretation in this classroom as well, there was no need to justify her solution. The teacher recorded this way on the inventory form and then asked for another way. At this point, the following exchange took place:

Bob: Um we could have 11 boxes, I mean 11 rolls.

Teacher: OK, what did you do to get 11 rolls?

Bob: Well, I unpacked a box.

Teacher: All right, so you are going to unpack a box.

Bob: Then you get 2 boxes, and then you get 11 rolls, and 6 pieces.

Teacher: OK, so Bob says that if we unpack a box, another way to have it would be 2 boxes, and then 11 rolls, and 6 pieces. Anyone have a question for Bob?

Teacher: ... Lenny?

Lenny: It would still be 3 boxes, 1 roll, and 6 pieces. Because this is 11 (points to numeral "11" in the rolls column of inventory form) and you take a ten from the 11, put it over here (points to boxes column), and make 3 and then you would have one more ten and that would make 1 and you still have 6 pieces.

Teacher: OK, could you show me in the storeroom what you are talking about doing?

At this point, Lenny came up to the screen as shown in Figure 4 and pointed to the ten rolls to be packed up.

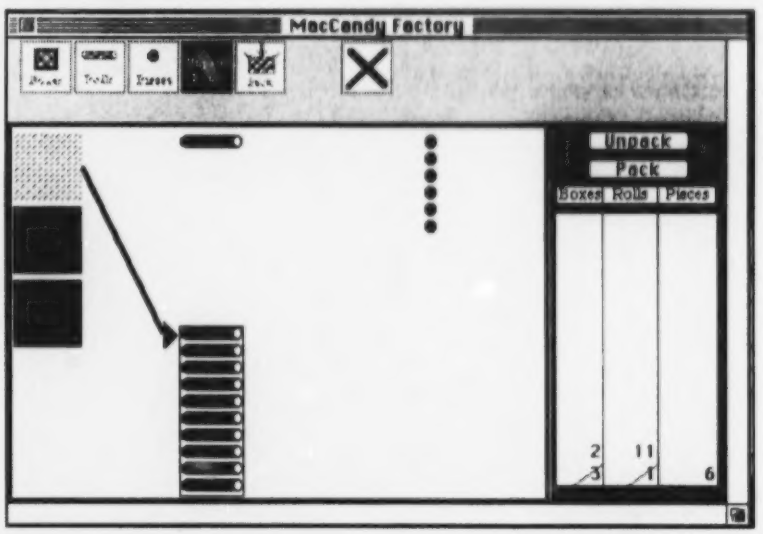


Figure 4. Screen capture showing Bob's suggestion to unpack a box.

After this exchange, the teacher asked for several more different ways. The students suggested unpacking the second and third boxes and then unpacking rolls. Throughout this discussion, the practice evolved such that each student framed his or her new way in terms

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of activity, rather than configuration. That is, each student began by stating, "I would unpack a [box, roll]." The teacher would unpack the box or roll using the graphics in the MacCandy factory, then ask how many boxes, rolls, and pieces would be in the storeroom after the transformation, and, finally notate this new state on the inventory form. At the conclusion of each transformation, the teacher would ask if there were any questions, and, if not, elicit another different way. Thus, in contrast to the preceding segment, where the prevailing means of justification involved "checking" each different way, no checking was involved. It appears that the class implicitly negotiated that there was no need for checking after Lenny's initial observation that each new way was a result of an unpacking that could be re-packed to get the same amount.

A second shift that occurred during this episode was that some students were beginning to anticipate the results of activity. For these students, the unpacked pieces were parts of the whole 316. For example, after the students discussed several ways to unpack rolls, Bob suggested the following transformation of the 31 rolls and 6 pieces that were displayed:

Bob: You keep on taking rolls off, like you take off another roll and you have 29, and then you go 28, 27, 26, 25...

Teacher: So, so what happens?

Bob: (comes to inventory form where teacher is writing) See, you would unpack all of these over here (points to numbers in rolls column) and move them all over to here and then you would have the same amount.

Bob's suggestion shifted the discourse from being rooted in anticipating single acts of unpacking to imagining what would happen if all of the rolls were unpacked. Like Lenny, Bob justified his suggestion by stating, "... you would have the same amount." His language indicates that, for him, rolls were composite units that could be counted as tens or single units. Further, activity with the numerals in the rolls column signified activity with the rolls for him. The action of manipulating numerals or computer icons was fused with the

imagery of acting with candies (cf. Nemirovsky, 1995). This action supported his reasoning, which we infer involved part-whole relations.

The teacher capitalized on Bob's insight by clicking on several more rolls to illustrate the action of unpacking, and also notated each of these transformations using the numerals on the inventory form. At this point, the students agreed that the screen appeared "messy." Several students including Lenny raised their hands to indicate that they did not understand the notation on the inventory form. The teacher then took this opportunity to relate the inventory form notation to the imagery of the candy factory. She cleared the screen and entered 30 rolls, 16 pieces, and asked Lenny if that was still the same amount, 316 candies. To answer this, Lenny came up to the front and counted by tens while pointing at each roll, and then counted-on by ones as he pointed to each individual candy icon.

Lenny: 10, 20, 30, 40, 50,60, 70, 80, 90, 100, that's one box. 10, 20, 30, 40, 50, 60, 70, 80, 90, 100, that's another box. 10, 20, 30, 40, 50, 60, 70, 80, 90, 100, that's another box. Then 310, 311, 312, 313, 314, 315, 316.

This counting by tens activity indicates that Lenny created composite units as he counted. It is interesting to note that he counted the graphics rather than reasoning with the numerals as he had previously done when verifying that 2 boxes, 11 rolls, and 6 pieces was the same amount as 3 boxes, 1 roll, and 6 pieces. We conjecture that for students such as Lenny who created composite units, the complex symbolic notation did not support their reasoning because the drawings were things to be counted whereas the numerals were merely a record of activity. Therefore, he explained his activity in terms of numerals, but did not explain the numerals in terms of activity. One observation from this exchange is that the linked representation system included in the microworld supported all students' efforts to make sense of their activity by either counting the graphics or reasoning with the numerals.

After Lenny verified that the 30 rolls and 16 pieces on the screen was the same amount as the original 316, the teacher asked Bob to explain his suggestion again. While Bob was talking, the teacher proceeded to unpack several rolls in succession using the

graphics in the MacCandy factory. Bob stated, "If we kept on doing that until we had all the pieces over here (pointing to the candies column of the inventory form), that would still be the same amount." This indicates that Bob was reasoning symbolically with the numerals on the inventory form.

The critical difference between these three episodes involves children's normative understanding of what constitutes justification for their different ways. As noted earlier, this difference illustrates the reflexivity between the constitution of sociomathematical norms and the variety of personal interpretations that are consequently made as students adjust their interpretations to changing aspects of the social setting. During the second episode, justification involved describing how each new configuration could be transformed into the original amount. For example, when Linda described packing up 500 pieces into five boxes, she justified her answer in terms of imagining activity. In contrast, justifications in the first and third episodes were more implicit. That is, instead of describing their ways of justifying with action imagery, students described how they constructed their different ways. For example, in Episode 1, students packed up rolls to make another box. Similarly, in Episode 3, students unpacked to make a new configuration.

This evolution illustrates how students developed conceptions of transformations as conserving quantity. Initially, each new way had to be completely packed up. Over the course of time as students reflected on their activity, they developed new conceptions of boxes and rolls as signifying the results of packing and unpacking. Thus, the norms for justifying became implicitly linked to students' reflections on the meaning of their activity. In this way, we claim that these norms were reflexively related to their conceptions and understandings. Further, this analysis is also consistent with the view that instructional materials cannot embody mathematical concepts. By analyzing how different materials supported qualitatively different discussions and emerging conceptions, these analyses attempt to describe the variety of issues and consequent reorganizations that occurred as the students worked in the different settings.

Evolution of Taken-As-Shared Mathematical Practices: Notating Transactions

So far we have discussed how children developed taken-as-shared meanings for the drawings and for transformations of quantity. The complexity of symbolizing and notating increased dramatically when the activities shifted from transforming quantities to solving problems that involved transactions. Transaction problems were of two general types. Either additional candies were made and brought into the storeroom or candies were taken out to fill an order. In either case, the problem was to figure out how many candies were in the storeroom after the transaction had taken place. From the observer's perspective, transaction problems could be viewed as multi-digit addition and subtraction problems. The critical issue that will be examined in this section is the reflexivity between the ways of notating transactions and the students' consequent reflections on their activity.

Symbolizing transaction problems requires a means of indicating quantities that have been sent out of the storeroom (or added to the storeroom) as well as a means of indicating packing and unpacking activity. In both teaching experiment classrooms it became important to develop a means of notating that distinguished between these two actions. For example in each classroom, it became apparent to all involved, the teacher, the children, and the researchers, that using the same symbol, such as an X, to indicate both unpacking a box (or roll) and to indicate sending out a box (or roll) was confusing. However this is precisely the type of issue that emerges from experiences in the classroom and illustrates the flexibility inherent in the teaching experiment methodology. As a result, notational meanings became an explicit topic of conversation in each classroom. For example, in the first teaching experiment, children working in pairs to solve transaction problems developed their own notational schemes and conventions. In the same lesson some children used circling to indicate taking away and crossing out to indicate unpacking while others used crossing out to indicate taking away. During the whole-class discussion that followed small-group work on these problems, these contrasting uses of cross-out notation were confounded with the

differences in the solution methods. As a result, the methods of notating became an explicit topic of conversation for the purpose of clarifying the "local" nature of these conventions. For example, Beauty explained how she and her partner solved the problem of unpacking to send out 162 candies from four boxes, 1 roll, and 9 candies, and figuring out how many were left, as follows:

Beauty: There was only one ten and they wanted 162 and there was only one ten and we took away five tens from one box and we got 5 tens left and we added it with the one ten and that was 60. So we circled the one box and that was 100 and then we took away those 60 and we circled two and we had seven left and that's how we did it.

With emphasis in her voice the teacher then said,

Teacher: Okay you *circled* the ones that you *took away*. So they circled one box. ...

Discussions such as these are significant because they make explicit the notational aspects of the problem. In this case the teacher made a distinction between the differences in the problem solution methods and in the means used to record and notate the solutions. The teacher's comments were explicitly about means of notating. In effect, the topic of discussion was the relevance of a standard notation. After this episode the teacher was, herself, much more careful to be consistent in her use of notation. In doing so, she initiated a standard means of notating, which then was shared by the class. It is possible that the need for a consistent notation emerged as an issue for the teacher through this experience. As this example illustrates, the means by which notation becomes taken-as-shared in a classroom are both explicit and implicit. In the case of the two classrooms we studied, each class eventually developed a standard means of notating transactions and transformations using drawings. Further, in the second, but not the first class, standardized means of notating transaction activity using numeric symbols only also became taken-as-shared.

In the remainder of this section we discuss the evolution of the classroom mathematical practices as children engaged in solving transaction problems. First we give an

overview of the evolution in the first teaching experiment classroom. Then we give an in-depth analysis of an episode from an interview in which a pair of students had access to the MacCandy Factory microworld to solve problems. This analysis clarifies children's differing interpretations and activity and relates them to their numerical reasoning.

Paper and Pencil Activity with Drawings. An initial phase in the use of drawings was that children operated on them to solve problems. That is, they carried out transformation and transaction activity with drawings and then determined the answer by figuring out the quantity indicated by their final configuration. Later, students used the visual material to support their problem-solving activity but did not rely solely on the drawings as the means of conceptualizing their answers. For example, to solve the problem shown in Figure 5, Beauty and Kati drew two boxes, five rolls, and two candies and began counting.

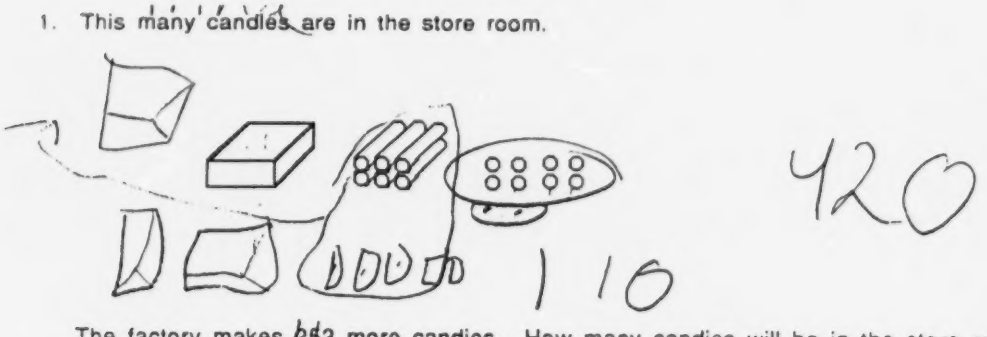

1. This many candies are in the store room.
420
110
The factory makes 252 more candies. How many candies will be in the store room now?

Figure 5. Scanned image showing Beauty's and Kati's problem solution.

Kati and Beauty: 100, 200, 300, (pointing to the boxes), 310, 320, ... 400 (pointing to the six rolls given and four of the rolls they have drawn).

Next they talked about making another box and drew an additional box on their paper. They then circled 10 rolls and drew an arrow to the additional box (see Figure 5). They continued.

Kati and Beauty: 10, 11, 12, 13, ... 20 (pointing to the remaining roll and to the individual pieces of candy).

Kati: Put these together (circling the 8 pieces of candy and the two pieces of candy) to make another 10. That's how we got 420.

In this example the girls did not, a priori, verbalize an intention of packing up the candies to determine the quantity. Rather they proceeded to determine the quantity by counting from their drawing. However, they interrupted their counting activity in progress to reconceptualize the quantity in terms of making another box. Further, after counting the remaining roll and the individual candies to get 20, Kati reconceptualized the ten candies as one roll. She both created the new units (box and roll) in the drawing and indicated through her explanation that she had created them as composite units. In fact, the drawings of the new units were made after she created the units conceptually. Nevertheless, it is apparent that the drawings supported Kati's solution activity and contributed to her reconceptualization of the units (ten candies as one roll and ten rolls as one box.)

As the small-group problem-solving episode continued, these girls continued to make advances in their understanding of composite units and of the imagery of the candy factory. At one point, Beauty explained that the drawings now served a symbolic function for her. She explained that she was erasing "my rolls because I can't draw good." Then instead of attempting to draw rolls iconically, as she had done up to this time, she simply used lines to represent rolls. In doing so, she gave explicit evidence that for her the drawings now were symbols. They were no longer pictures of rolls.

At the same time, Kati began describing her problem solutions purely abstractly, indicating that for her the imagery of the candy factory may have been dropping out. At a minimum, she was developing ways to partition 100. For example, to solve the problem shown in Figure 6, Kati circled one box and then said, "Take 51 out of there so there's 49 left." Here we infer that Kati was not thinking of unpacking a box to get 10 rolls. Instead, she was thinking of a box as 100 and either "just knew" that 51 and 49 make 100 or else was

able to figure it out very quickly. It might be argued that Kati's concept was a unitary one. To solve the problem as she did, did not require that she conceptualize the box as ten units of ten. All that was needed was to think of it as 100 units of one. However, because students in this class did not work at developing partitions of 100, we infer that her knowledge of such partitions developed through working with the problems in the candy factory scenario.

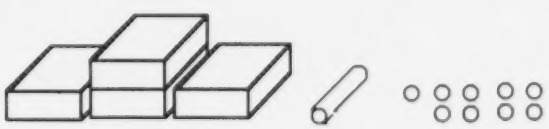
This many candies are in the store room.

How would you unpack some of these candies so that you would be able to send 151 candies to a shop. How many candies would be left in the store room?

Figure 6. Problem solved by Kati using a partitioning of 100 strategy.

Other students as well gave indication that the imagery of the candy factory was functioning symbolically for them. For example, consider the whole-class discussion of the problem shown in Figure 7.

This many candies are in the store room.



How would you unpack some of these candies so that you would be able to send 162 candies to a shop. How many candies would be left in the store room?

Figure 7. Problem solved by Teri and Shomekia in different ways.

After one solution was explained the teacher asked if anyone solved it a different way. Shomekia, one of the children who was still operating on the drawings to solve problems, offered the solution she and her partner developed as different because they had "opened the box on the right" whereas in the previous solution the teacher marked a different box as the one to be opened. Subsequently, another pair explained that they opened a box and took out 60. When the teacher asked, "You opened up a box, which box?" Teri replied, "Any box." In replying in this way, Teri indicated that for her the drawings were images to think with. She did not have to have a particular box that she had opened. Even though the written work on Teri's and her partner's paper showed a specific box as the one opened, her comment that "any box" could be opened indicates that for her the written work served as a means to record her imagined activity. She did not manipulate the drawings to arrive at her solution. Teri's solution activity stands in stark contrast to that of Shomekia who operated on the drawings and for whom the drawings represented specific collections of boxes, rolls, and candies. We note that these contrasting solution methods illustrate a key feature of the Candy Factory instructional sequence. The instructional tasks are designed so that they can be completed meaningfully by children at a variety of different conceptual levels. Clearly, the drawings had very different meanings for Shomekia and for Teri.

Mental Problem Solving with Visual Support Materials. As the instructional sequence progressed and the children advanced in their ability to coordinate units of different rank, the way they made use of the imagery of the candy factory also evolved. We have shown that initially students operated on drawings to generate solutions to problems. Later, the drawings were a means of supporting the children's thinking and contributed to their conceptualization of quantities but were not manipulated to generate answers. For example, Kati created composite units conceptually first and only later showed them in her drawings. A further advance occurred when instructional tasks were presented showing some, but not

all, of the quantities in drawings on the overhead screen and children worked without paper and pencil using only the overhead projection of the drawings and the imagery of the candy factory to solve problems. In this case, students were unable to physically manipulate the drawings in any way. All of their activity had to be conceptual.

For example, the teacher flashed a drawing of four boxes, seven rolls, and five candies on the left side of the overhead screen for several seconds (see Figure 8). She flashed it again briefly a second time. With the overhead projector turned off, the children discussed the quantity of candies they had seen. As the discussion progressed the teacher displayed the image to facilitate the dialogue. After the children agreed that the quantity shown was 475, the teacher left the overhead material visible and told the students that there were an additional 52 candies hidden under a cover displayed on the right side of the overhead screen. The teacher wrote the numeral 52 on the overhead for the children to see.

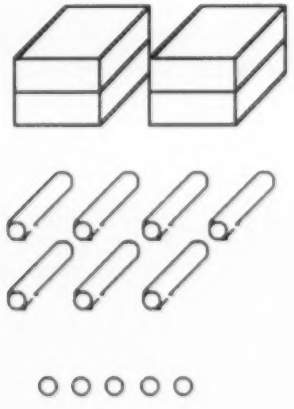


Figure 8. Visual image of 475 shown on the overhead screen.

The question posed was, "How many are there altogether?" To solve this problem children had to create their own mental images of 52 or operate in more symbolic ways with the quantities.

The explanations that children gave when describing their solutions for tasks of this type typically involved a to-ing and fro-ing between the language of quantity and the language of the candy factory suggesting that they were able to create mental images of the quantity that was described only numerically which they could use in conjunction with the visual material displayed. For example, Teri gave the following explanation:

Teri: I saw four boxes. And it was 52 and then I took that 50 off of the two and I added that 50 on to the 7. And one of the 10s from the fifty and I got 80 and another one and I got 90 and another one, I got 100. So I said I had another box. ... I knew I had two 10s left and that 5 ... I said 510, 525. I said two left from that 52. I said 527.

We infer from this explanation that Teri was able to think of the 52 in terms of five 10s and 2 ones or in terms of rolls and candies. These seemed to be interchangeable for her and indicates part-whole reasoning. Teri could be described as reasoning above her activity and reflecting on it. Further evidence is given by the re-explanation she gave in response to another student's request to clarify whether she was talking about 10s or rolls.

Teri: ... 7 rolls, I knew it'd be 70 in all. So I said, 70, 80, 90, 100. I knew that was 3 rolls. I knew it was 30 so I added the 30 to the 7 rolls and got a box.

In referring interchangeably to 70 and 7 rolls and 30 and 3 rolls, Teri is demonstrating the type of flexibility that Steffe talks about when he says that children need to be able to coordinate units of different rank to have an understanding of place-value numeration (Steffe, 1992). The significant aspect of this example is that the quantity 52 was described only in terms of numerical symbols. Thus, we can conclude that for Teri, these numeric symbols served the function of symbolizing a complex conceptual web of relationships, images and understandings. She could use a variety of images and relationships

interchangeably, at will, as she operated with the quantity 52 and the quantity 475 represented by the visual material.

Certainly, not all children in the class were able at this time to operate in the sophisticated way that Teri did. However, as they participated in class discussions such as these they had the opportunity to listen to and attempt to make sense of solutions such as Teri's. Furthermore, they were under the obligation to ask questions and seek clarification when they did not understand (Underwood-Gregg, 1996). In the course of the discussions, the teacher assisted by elaborating the explanations children gave, by initiating discussions that focused on comparing and contrasting solutions and by relating more formal solutions to the candy factory imagery. As a result each child was in the position of being able to advance their own understandings.

One way to describe the goal of the Candy Factory instructional sequence is that children will construct a deep understanding of place-value numeration by continually reorganizing their imagery of acting with collections. This implies that they are able to interpret multi-digit numerals and operate on them meaningfully in addition and subtraction situations. The Candy Factory scenario is intended to provide an imagistic basis that can first function as a *model of* their activity and later serve as a *model for* more formal reasoning (Cobb & Yackel, 1993). In this regard, the inventory form was intended to provide symbolic support for the children's activity without providing accompanying visual imagery. In the second classroom teaching experiment much attention was given to developing the inventory form as a notational means that shows a record of packing, unpacking, and transaction activity. However, as we have noted earlier, for a variety of reasons this aspect received little emphasis in the first classroom. Instead, children typically operated mentally with little notational support, or made drawings to solve problems posed numerically. For example, in a whole-class discussion the teacher posed the following problem situation.

> The storeroom manager, Mr. King, had 572 grape candies in the storeroom. He sent out an order for 43 grape candies. Mr. King wants to know how many grape

candies are in the storeroom now. However, he does not want to start counting them all over again. So he decided to take away 43 from his previous total. How many grape candies were there in the storeroom?

The teacher used the following notation to show the problem on the overhead screen.

grape

| | | | |
|---|---|---|---|
| | 5 | 7 | 2 |
| – | | 4 | 3 |
| | | | |

Teri: I think its 539. I took 40 from that 70 and I had 30 left and I took that 2, and I opened a ten pack from the 30 that I had left and I took one from out of there and then I had 9 left plus—I changed my answer to 529.

The evolution in the type of reasoning that has taken place over the course of the instructional sequence is that now Teri is operating completely with the numeric symbols. Yet, she gives one hint that the imagery of the candy factory is still useful in supporting her thinking, when she says, "I opened a ten pack." In redescribing her solution the teacher cast it entirely in terms of the candy factory as a means of supporting other children's attempts to make sense of Teri's explanation.

Teacher: You took four rolls away from 7 rolls and you came up with 3 or 30. Right? You had 530. Then you took one of those rolls and you opened it up. Is that right?

Teri nodded in agreement. Whether or not Teri thought of the 70, the 40, and the 30 as 7, 4, and 3 rolls, respectively is not clear. However, for her the candy factory imagery was consistent with her solution and so she agreed with the teacher's reformulation using the language of the candy factory. The problem was posed in terms of activity in the candy factory but the numeric notation placed it within formal, conventional mathematical notation. As this was the first lesson where transaction problems were introduced using the inventory

form, the teacher had not yet worked out a means of notating actions such as Teri's. For several lessons both she and several students began using a crossing out notation reminiscent of conventional carrying and trading notation. However, as we have noted earlier, for a variety of reasons, a consistent means was not developed in this classroom.

As a final point, we note that notational schemes not only facilitate solution activity, they also constrain it. Constraints can be beneficial depending on one's goals because they delimit the possibilities. In this case, if the goal is to help children develop meaningful interpretations of the conventional notation that accompanies traditional algorithms, then it might be useful to discourage certain solution methods which do not lend themselves to such notation. At a minimum, certain solution methods that are accessible only to those who can coordinate units of three different ranks with little or no symbolic support may not be productive for other students whose understandings are less sophisticated. For example, in solving the problem of sending out 253 candies from a supply of 541, one pair of students said that they sent out two boxes. Then they opened up one of the three remaining boxes and took out 53 candies and added the remaining 47 candies to the 41 candies and to the 200 get a total of 288. In our experience this type of solution is given only by those children who have a complete understanding of place-value numeration. We would argue that it would not be productive for the majority of the class for the teacher to develop notational schemes specifically for solutions of this type. Rather, it would be more productive to develop a conventional scheme and leave solutions such as this to go unnotated. We speculate that if this were to happen, such solutions would be given less often. Students are quick to discern what it is that is valued in the classroom (McClain, 1995; Yackel & Cobb, 1996).

Shifts in Problem Solving with the MacCandy Factory. The variety of conceptual understandings that arose in the first classroom (as described in the previous section) also arose in the second classroom. In the following episode, Crystal, like Teri, appeared to reason in terms of part-whole relations. Both could reason symbolically with numerals, both

could anticipate activity, and both appeared to view drawings as records of activity. In contrast, Crystal's partner, Martine, appeared to create composite units to support his thinking as Beauty did in the previous segment. In this episode, Crystal and Martine are working with the MacCandy Factory microworld to solve the following problem:

Problem: *In the storeroom at the candy factory there are 504 candies. Mr. Strawberry sends out 69 pieces. How many are in the storeroom now?*

Initially, Martine solved the task by writing it as a vertical subtraction problem by writing $\begin{array}{r} 504 \\ -69 \\ \hline 194 \end{array}$. He explained his answer as follows:

Martine: 194. What I did is, I wrote down the problem then I took away, there was nothing down there, I put the 4 like there was a zero right there, and then I just put 9 right there. And then 5 plus 6 is 1 [sic].

In response, his partner, Crystal, stated, "When he wrote that down um, he said that there were 600, 6 boxes, and on the sheet it says they only made 69 pieces, so, he should have the 9 ..."

Martine then revised his work as shown in Figure 9.

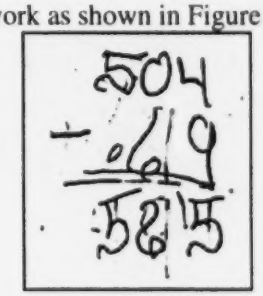



Figure 9. Scanned image of Martine's revised work.

There are several interesting aspects of Martine's interpretation. First, he did not interpret the 504 or the 69 in terms of boxes, rolls, and pieces. Instead, he appeared to interpret the problem as involving "504 pieces" in the storeroom and "69 pieces" in the order. Thus, for him, the two quantities were numerical composites, which is consistent with his interpretation of the "500 rolls" discussed in a previous section and also appears to

resemble Kati's interpretation when she solved the problem shown in Figure 6 by partitioning 100.

Crystal's interpretation of Martine's work indicates that, for her, the placement of the digits did indicate that they signified the number of boxes, rolls and pieces as they would on an inventory form. This is implicit in her comment, "When he wrote that down, um, he *said* that there were 600, 6 boxes..." Even though Martine never *said* anything, Crystal interpreted his notation as indicating a written communication. From her point of view, the symbols had quantitative significance. That is, her explanation to Martine was not simply an arbitrary rationale based on the mathematical practices of the classroom (in which digits represented boxes, rolls, and pieces respectively). Instead, she, like Teri, viewed numerical notation as a way to record activity.

As the students began to work with the microworld, Crystal stated that she wanted to unpack a roll, while Martine stated that she needed to unpack 7 rolls. This indicates that for Martine, rolls were composite units, whereas for Crystal part-whole relations were involved. For her, the 6 rolls were part of one unpacked box, and 9 pieces were part of one unpacked roll. Because these different interpretations never become explicit topics of conversation, the two talked past each other as they attempted to negotiate which rolls to unpack. For example, Crystal unpacked one roll. To her, this was sufficient to send out 9 pieces. However, Martine still maintained that Crystal needed to unpack 7 rolls. For Crystal, the tricky part of this problem involved actually *doing* the sending out that she envisioned (i.e., figuring out which buttons to press to send out). In contrast, the tricky part for Martine involved envisioning how this order could be sent out given the present state of the inventory form (see Figure 10). Therefore Crystal could be described as reasoning above her activity and reflecting on it, while Martine was in his activity and could not see beyond his local interpretation.

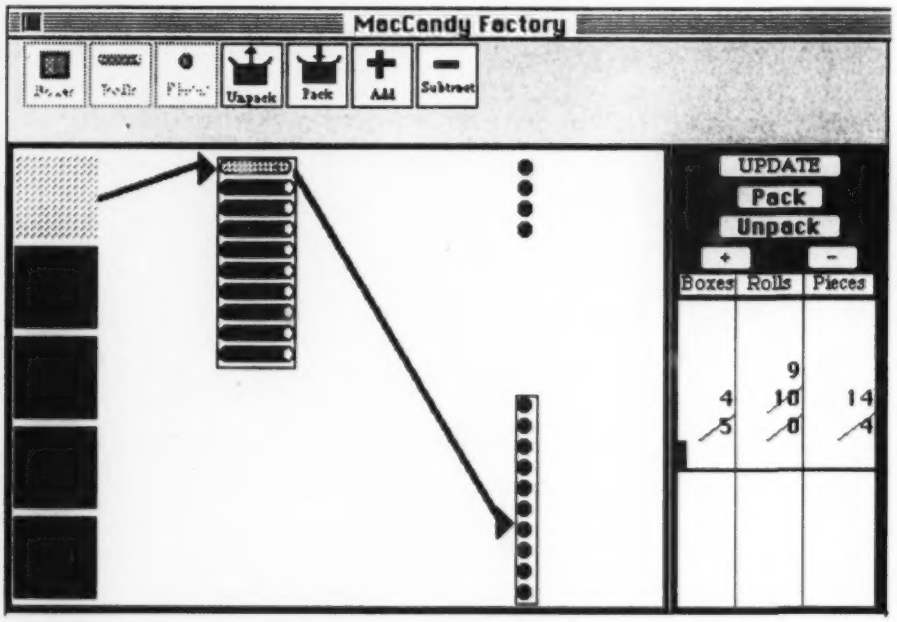



Figure 10. Screen capture showing unpacked box and unpacked roll.

After Crystal sent out 9 candies, she and Martine sent out 6 rolls. Thus, Martine appeared to reconceptualize his activity after he and his partner sent out 9 pieces. From an outside observer's perspective, the activity that transpired during the final part of this episode might be described as follows: the students figured out how to symbolize sending out; clicked on 6 icons representing rolls, and 9 icons representing pieces; and then re-presented their activity on the inventory form. However, from the students' point of view, they might explain it as follows: "First we sent out pieces and rolls, and then we counted the number of candies, boxes, and rolls that were left." These differing interpretations illustrate two points. First, the students did not view their activity with the inventory form as re-presenting their activity with the graphics as the research team had intended. Instead, it was constituted as a recording device. Second, from these students' perspective, acting with the graphics *was* acting with the candies, it did not simply represent it. In other words, acting with the graphics was fused with the imagery that it signified (cf. Nemirovsky, 1995).

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This notion of fusion does not suggest that both students interpreted their activity in the same way. For Crystal, the activities of clicking on graphics and clicking on the numerals in the inventory form were all interrelated. The implication from this perspective was that she, like Teri, could reflect on both of them in the same way. However, for Martine, who was following Crystal's instructions, the form was constituted as a place to record the results of acting on the candies. This parallels Shomeka's conception of drawings as things to act on. That is, for Martine, the inventory form and the graphics may have been empirically related, but he did not appear to have reinterpreted his pencil and paper work as a record of activity. However, we infer that acting with the dynamic symbols may have supported his efforts to reorganize his conceptions of ten.

## Conclusion

In this paper we have investigated the mathematical meanings that emerged as children participated in the Candy Factory instructional sequence, with specific emphasis on the role of imagery and symbols. We have attempted to show that the conceptual understandings the individual children developed and the classroom mathematical practices that emerged were both constrained and enabled by the meanings of the pedagogical symbols that were negotiated in the teaching experiment classrooms. For example, the dynamic nature of the symbols in the MacCandy Factory supported conceptual discourse that focused on acting with collections. This interpretation, which was also supported by the dynamic use of the overhead flats, longs, and cubes, contributed to the classroom practice of justification by packing and unpacking. By contrast, a notion of equivalence based on total quantity, which predominated in the example with the chart of iconic drawings, was more likely to emerge when static symbols were used. As the examples show, some children did develop arguments to justify equivalence based on packing and unpacking in static situations. However, in the case of dynamic symbols of the MacCandy Factory microworld, the students developed ways of anticipating activity on collections and notating changes with numerals as well as with graphics. In this way the conceptual basis for conventional column

notation involving trading was supported. At the same time, the built-in system for numeric notating on the inventory form constrained interpretations of quantities as composite units, such as Martine gave when he wanted to send out 69 candies by unpacking 7 rolls, by making them inefficient to use.

We have also attempted to show how the imagery of the candy factory scenario itself, coupled with visual support material, such as drawings and overhead materials, contributed to children's understanding of place-value numeration. Initially the Candy Factory scenario supported their informal activity and their efforts to draw transactions and transformations, such as when Beauty and Martine solved problems by manipulating drawings and counting the quantity in the final configuration. Later, at least for some students, such as Teri and Crystal, these drawings came to signify their more formal reasoning, as when they solved problems numerically. As our examples show, children's conceptual understandings are not uniform within a class. Thus, at any given time some children's interpretations may be at variance with what is taken-as-shared by the class. Class discussions in which these interpretations are explored are powerful learning situations because they make explicit what is otherwise only implicit. Throughout this paper, we have attempted to show how explicit discussions that relate the imagery of the candy factory, quantity, and various symbolic and notational schemes contributed to the emergence of mathematical meaning for place-value numeration.

## Educational Importance

The implications of this study reach beyond an analysis of the children's activity in the immediate instructional sequence. The study illustrates how the evolution of various mathematical practices supported students' development of increasingly sophisticated mathematical conceptions, for example, how students' use of symbolization to record and communicate their actions may support their development of place-value conceptions. Further, the role of the computer microworld in supporting children's conceptual development and in facilitating communication in classroom discussions was also

demonstrated. For example, by including both graphics and numerals, everyone had a way to participate in the discussions. This also enabled the teacher and some students to capitalize on the dynamic quality of computer graphics to support descriptions of their more formal activity with the imagery of the underlying scenario, in this case, the Candy Factory.

Finally, this analysis also highlights the importance of assuming a bottom-up perspective in which the students' individual conceptions are seen as emerging from participation in classroom activity. The variety of mathematical conceptions that arose in the two classes indicate that neither materials, nor computer programs, nor instructional sequences can be taken as sufficient. The reflexivity between students' emerging conceptions and their participation in classroom practices involves constant interpretations and reinterpretations on the part of the students, teachers, and researchers.

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